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THE FORMULATION OF TRANSIENT AXISYMMETRIC BOUNDARY-VALUE PROBLEMS OF THERMOELASTICITY FOR SHELLS OF REVOLUTION AND ALGORITHMS FOR SOLVING THEM†

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Algorithms for solving boundary-value problems and for computing temperature fields and thermal stresses are considered for a certain class of structures whose main element is a thin-walled shell of revolution subject to external pressure under general conditions of unsteady heat exchange with the environment. Within the framework of Meissner's computational scheme [1], a system of differential equations is obtained for the axisymmetric bending of arbitrary shells of revolution, using a linear coordinate along an arc of the meridian. For the joint and simultaneous solution of these equations, with a calculation of the temperature fields in meridional sections of the shell, the heat-conduction equation is obtained in a similar coordinate system with a curvilinear coordinate s along a generator and a coordination y along the normal to the shell surface. Algorithms, obtained using the finite-difference matrix double-sweep method [2–4], are proposed for the practical solution of boundary-value problems to compute the unsteady temperature fields and stresses. © 2004 Elsevier Ltd. All rights reserved.

1. THE DIFFERENTIAL EQUATIONS OF THE BENDING OF SHELLS UNDER AXISYMMETRIC EXTERNAL PRESSURE AND HEATING

A computation of the temperature stresses for shells of revolution under conditions of unsteady heat exchange with the environment involves the simultaneous solution of two boundary-value problems. One of them is formulated and solved to compute the temperature fields, but in order to compute the thermal stresses one has to determine the strains, internal forces and moments of elastic interaction of the elements of the shell when there is a non-uniform variation of the temperature over its thickness and generator.

If it is assumed that the strains of the shell when there is an axisymmetric application of heat are non-flexural, the computational formulae for the meridional and circumferential stresses, σ_m and σ_φ, respectively, are obtained in the same way as for a thin-walled cylinder, i.e.

$$\sigma_m(y,s) = \sigma_\varphi(y,s) = \frac{E\alpha}{1-\mu}(T_a(s)-T(y,s)), \quad T_a(s) = \frac{1}{h}\int_{-h/2}^{h/2} T(y,s)dy \tag{1.1}$$

where $T(y, s)$ is the distribution function of the temperature over the shell thickness and the meridional coordinate s at a given time t, y is the coordinate along the shell thickness, directed along the outer normal from a point on the middle surface, h is the thickness, μ is the Poisson's ratio, α is the coefficient of thermal expansion and E is the modulus of elasticity.

In the zones characterized by sudden or continuous, but very rapid, changes of temperature along the generator, and also in the neighbourhood of the fixed end and the boundaries of variation of the shapes and dimensions of the component shells, bending strains arise, if these are taken into consideration, the formulae for computing the stresses may be written in the form

$$\sigma_m(y,s) = \frac{E\alpha}{1-\mu}(T_a(s)-T(y,s)) + \left\{\frac{N_m(s)}{h} + y\left(\frac{d\vartheta}{ds} + \mu\frac{\operatorname{ctg}\theta}{R_2}\vartheta(s)\right)\right\} \tag{1.2}$$

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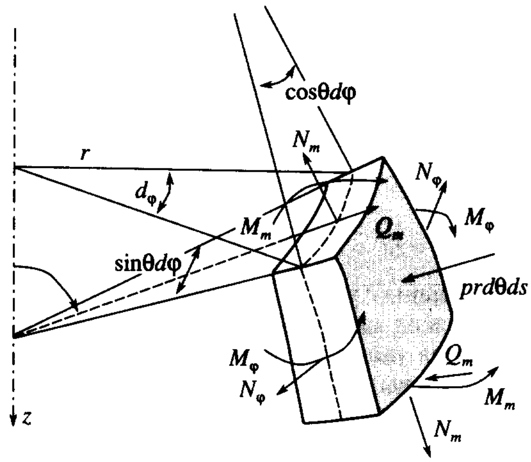

Fig. 1

$$\sigma_\varphi(y, s) = \frac{E\alpha}{1-\mu}(T_a(s) - T(y, s)) + \left\{\frac{N_\varphi(s)}{h} + y\left(\frac{\operatorname{ctg}\theta}{R_2}\vartheta(s) + \mu\frac{d\vartheta}{ds}\right)\right\} \tag{1.3}$$

where $\vartheta(s)$ is a function representing the angles through which the normal to the middle surface of the shell rotates during bending, and N_m and N_φ are the meridional and circumferential internal forces of the shell.

Thus, in order to determine the stresses, taking the bending strains into account, one must determine the internal forces and the function $\vartheta(s)$ representing the angles through which the normal rotates. The internal forces of the shell N_m and N_φ may be expressed in term of the function of the shearing forces $Q_m(s)$. That is done using two equations. One of them is the equilibrium equation of the projections of all forces onto the axis of revolution of part of the shell cut out by a circumferential section. For shells closed at the vertex it has the form

$$2\pi r(N_m \sin\theta + Q_m \cos\theta) + \int_0^s p\cos\theta 2\pi r ds = 0 \tag{1.4}$$

where θ is the angle between the normal to the shell surface and the axis of revolution, $r(s)$ is the radii of the parallels of the middle surface and $p(s)$ is the external pressure.

The second equilibrium equation is set up for a small element of the shell cut out by two circumferential and two meridional sections, as shown in Fig. 1. The condition that the sum of projections of all forces onto the normal must vanish implies that

$$\frac{1}{r}\frac{d}{ds}(Q_n r) + \frac{N_m}{R_1} + \frac{N_\varphi}{R_2} + p = 0 \tag{1.5}$$

where R_1 and R_2 are the radii of curvature of the shell.

Equations (1.4) and (1.5) can be used to determine the meridional and circumferential internal forces and to express them in terms of the external pressure and the shearing forces function $V = R_2 Q_m$

$$N_m = -\frac{\operatorname{ctg}\theta}{R_2}V - \frac{1}{R_2}f(s), \quad f(s) = \frac{1}{\sin^2\theta}\int_0^s pr\cos\theta ds \tag{1.6}$$

$$N_\varphi = -\frac{dV}{ds} - pR_2 + \frac{1}{R_2}f(s) \tag{1.7}$$

These expression were derived using the relations $r = R_2\sin\theta$, $ds = R_1 d\theta$.

Since the internal forces are determined from the equilibrium equation, it follows that the shearing forces function must satisfy the strain compatibility equation. To derive that equation, we write out the expressions for the strains of the middle surface and angle of rotation of the normal

$$\varepsilon_m = \frac{du}{ds} + \frac{w}{R_1}, \quad \varepsilon_\varphi = \frac{w}{R_2} + \frac{u}{R_2}\operatorname{ctg}\theta, \quad \vartheta = \frac{u}{R_1} - \frac{dw}{ds} \tag{1.8}$$

where u, and w are the displacements of the points of the middle surface with respect to the meridian and the outer normal, respectively.

Transforming these expressions with the help of the third of them, we can eliminate the variables u and w and obtain the strain compatibility equation, for example, in the following form

$$\frac{d\varepsilon_\varphi}{ds} - (\varepsilon_m - \varepsilon_\varphi)\frac{\operatorname{ctg}\theta}{R_2} + \frac{\vartheta}{R_2} = 0 \tag{1.9}$$

If the strains are now expressed in terms of the internal forces, we obtain, using Eqs (1.6) and (1.7)

$$\varepsilon_m = \frac{1}{Eh}\left(-\frac{\operatorname{ctg}\theta}{R_2}V + \mu\frac{dV}{ds} - \left(\frac{1}{R_2} + \frac{\mu}{R_1}\right)f(s) + \mu p R_2\right) + \alpha T_a$$

$$\varepsilon_\varphi = \frac{1}{Eh}\left(-\frac{dV}{ds} + \mu\frac{\operatorname{ctg}\theta}{R_2}V + \left(\mu\frac{1}{R_2} + \frac{1}{R_1}\right)f(s) - pR_2\right) + \alpha T_a$$

Substituting these expressions into Eq. (1.9) and performing the necessary reduction, we obtain the first differential equation for the two unknown functions V and ϑ

$$\frac{d^2V}{ds^2} + \frac{\operatorname{ctg}\theta}{R_2}\frac{dV}{ds} - \frac{\operatorname{ctg}^2\theta}{R_2^2}V + \frac{\mu}{R_1R_2}V - \frac{Eh\vartheta}{R_2} = Eh\alpha\frac{dT_a}{ds} + \frac{1}{R_2}\Phi(s) \tag{1.10}$$

$$\Phi(s) = f(s)\left[\left(1 - \frac{R_2^2}{R_1^2}\right)\frac{\operatorname{ctg}\theta}{R_2} + \frac{d}{ds}\left(\frac{R_2}{R_1}\right)\right] - \frac{d}{ds}(pR_2^2) \tag{1.11}$$

The function $\Phi(s)$ was obtained after rather complicated computations, in the course of which the following relations were derived and used

$$\begin{aligned} &\frac{d}{ds}(pR_2^2) = 2pR_2\operatorname{ctg}\theta\left(1 - \frac{R_2}{R_1}\right) + R_2^2\frac{dp}{ds}, \quad \frac{d}{ds}\frac{1}{R_2} = \frac{\operatorname{ctg}\theta}{R_2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \\ &\frac{d}{ds}f(s) = pR_2\operatorname{ctg}\theta - \frac{2\operatorname{ctg}\theta}{R_1}f(s), \quad \frac{d}{ds}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{d}{ds}\frac{1}{R_2}\left(1 + \frac{R_2}{R_1}\right) \end{aligned} \tag{1.12}$$

To derive the second equation, one has to use the equilibrium condition for the moments of all internal forces shown in Fig. 1

$$\frac{dM_m}{ds} + \frac{\cos\theta}{r}(M_m - M_\varphi) - Q_m = 0 \tag{1.13}$$

where M_m and M_φ are the bending moments, which may be expressed in terms of the function of the angles of rotation of the normal in bending

$$\begin{aligned} &M_m = -D\left(\frac{d\vartheta}{ds} + \mu\frac{\operatorname{ctg}\vartheta}{R_2}\vartheta\right) + m, \quad M_\varphi = -D\left(\frac{\operatorname{ctg}\theta}{R_2}\vartheta + \mu\frac{d\vartheta}{ds}\right) + m \\ &D = \frac{Eh^3}{12(1-\mu^2)}, \quad m = \frac{E\alpha}{1-\mu}\int_{-h/2}^{h/2} T(y,s)y\,dy \end{aligned} \tag{1.14}$$

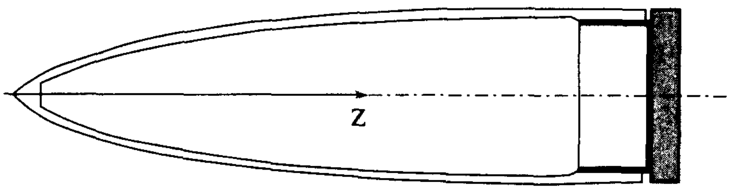

Fig. 2

Substituting expression (1.14) into the equilibrium equation (1.13), we obtain the second differential equation for the two unknown functions

$$\frac{d^2\vartheta}{ds^2} + \frac{\mathrm{ctg}\,\theta}{R_2}\frac{d\vartheta}{ds} - \frac{\mathrm{ctg}^2\theta}{R_2^2}\vartheta - \frac{\mu}{R_1 R_2}\vartheta + \frac{V}{DR_2} = \frac{1}{D}\frac{dm}{ds} \tag{1.15}$$

If only the thermal stresses are being computed, then $\Phi(s) = 0$ and the right-hand sides of the system just obtained are considerably simplified, while formulae (1.2) and (1.3) for computing the stresses become

$$\sigma_m(y) = \frac{E\alpha}{1-\mu}(T_a - T(y)) + \left\{-\frac{\mathrm{ctg}\,\theta}{hR_2}V + \frac{E}{1-\mu^2}y\left(\frac{d\vartheta}{ds} + \mu\frac{\mathrm{ctg}\,\theta}{R_2}\theta\right)\right\} \tag{1.16}$$

$$\sigma_\varphi(y) = \frac{E\alpha}{1-\mu}(T_a - T(y)) + \left\{-\frac{1}{h}\frac{dV}{ds} + \frac{E}{1-\mu^2}y\left(\frac{\mathrm{ctg}\,\theta}{R_2}\theta + \mu\frac{d\vartheta}{ds}\right)\right\} \tag{1.17}$$

This computational scheme was apparently first used by Meissner [5], but the derivation of the analogous differential equations involved the introduction of the angular coordinate θ, so that it is impossible to apply them to shells with conical and cylindrical sections. Boyarshinov [6, p. 402] derived the differential equations on the assumption that R_1 = const. Here, no restrictions were imposed on the radii of curvature in deriving the differential equations. Consequently, they may be applied to composite shells of revolution of any shape.

It has been shown in [6] that the necessary boundary conditions for the functions ϑ and V can be obtained for all the main methods of attaching the edges of the shell, as well as the conditions for the rigid coupling of two shells. If the shell is closed at the vertex and is of constant thickness h, then at $s = 0$ necessarily $\vartheta = 0$, $V = 0$.

Figure 2 illustrates a cusped shell with a supporting frame in the base. If the nose is massive, it must be isolated by a circumferential section with coordinate $s = s_0$, and a condition restricting the angles of rotation of the normal is imposed on the boundary at $s = s_0$; in addition, the circumferential strain of the shell must equal the thermal strain for equal temperatures of the shell and the massive nose at their common boundary

$$\vartheta = 0, \quad \frac{dV}{ds} - \mu\frac{\mathrm{ctg}\,\theta}{R_2}V = \left(\frac{1}{R_1} + \frac{\mu}{R_2}\right)f(s) - pR_2 \tag{1.18}$$

For elastic interaction of the shell and the frame, the condition for their circumferential strains to be equal, and the equilibrium equation of the shearing forces of the shell and the circumferential forces of the frame must be satisfied. The latter can also be expressed in terms of the unknown functions ϑ and V. As a result, the boundary conditions at $s = L_s$, where L_s is the length of the shell along the meridian, take the form

$$\vartheta = 0, \quad \frac{1}{Eh}\left(-\frac{dV}{ds} + \mu\frac{\mathrm{ctg}\,\theta}{R_2}V\right) + \alpha T_a = \varepsilon_t + \frac{1}{B}V$$
$$\varepsilon_t = \frac{1}{B}\iint_S E\alpha T dS, \quad B = \iint_S E DS \tag{1.19}$$

where ε_t is the thermal strain of the free frame and S is its area of cross-section.

As obvious from conditions (1.19), the circumferential strain of the shell turns out to be equal to the strain of the free frame in the limiting case, when the stiffness B of the frame to expansion tends to infinity, while when $\varepsilon_t = 0$ and $B \to \infty$ boundary conditions (1.19) correspond to the conditions of a rigid fixed end.

2. AN ALGORITHM FOR THE FINITE-DIFFERENCE SOLUTION OF THE BOUNDARY-VALUE PROBLEM OF COMPUTING THE STRESSES FOR A SHELL OF REVOLUTION IN THE CASE OF UNSTEADY HEATING

The problems under consideration, of computing the thermal stresses for shells of revolution in the case of high-speed external heating, are of topical interest for the design of many mechanical engineering structures. In that connection one is justified in considering a fairly simple and effective method for their numerical solution, which is accessible for use over a broad range of engineering practice – first and foremost, a finite-difference matrix double-sweep method, whose possibilities are presented below.

We introduce a dimensionless coordinate x along the arc of the meridian, dimensionless radii of curvature and new unknown functions F and Q of the same dimension in Па

$$x = \frac{s}{R}, \quad F = \frac{E}{\beta}\vartheta, \quad Q = \frac{1}{hR}V, \quad r_1 = \frac{R_1}{R}, \quad r_2 = \frac{R_2}{R}, \quad \beta = \frac{12(1-\mu^2)R^2}{h^2} \tag{2.1}$$

where R is a constant with the dimensions of length.

We represent the system of differential equations (1.10), (1.15) in vector-matrix form

$$\frac{d^2\psi}{dx^2} + \frac{\operatorname{ctg}\theta}{r_2}\frac{d\psi}{dx} - \frac{\operatorname{ctg}^2\theta}{r_2^2}\psi + C\psi = q \tag{2.2}$$

$$\psi = \begin{Vmatrix} F \\ Q \end{Vmatrix}, \quad C = \begin{Vmatrix} -\dfrac{\mu}{r_1 r_2} & \dfrac{1}{r_2} \\ -\dfrac{\beta}{r_2} & \dfrac{\mu}{r_1 r_2} \end{Vmatrix}, \quad q = \begin{Vmatrix} q_1 \\ q_2 \end{Vmatrix}, \quad q_1 = \frac{1}{hR}\frac{dm}{dx}, \quad q_2 = E\alpha\frac{dT_a}{dx}$$

A finite-difference analogue of this equation, using a standard three-point central difference scheme for the first and second derivatives, may be written as

$$A^-\psi_{k-1} - B\psi_k + A^+\psi_{k+1} = d, \quad k = 2, 3, \dots, N$$

$$A^{\pm} = \left(1 \pm \frac{\Delta x \cdot \operatorname{ctg}\theta_k}{2r_2(x_k)}\right)\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad B = \begin{Vmatrix} b^+ & c^- \\ c^+ & b^- \end{Vmatrix}, \quad d = \begin{Vmatrix} d_1 \\ d_2 \end{Vmatrix} \tag{2.3}$$

$$b^{\pm} = 2 + \frac{(\Delta x)^2}{r_2^2(x_k)}\left(\operatorname{ctg}^2\theta_k \pm \frac{\mu r_2(x_k)}{r_1(x_k)}\right), \quad c^{\pm} = \pm\frac{(\Delta x)^2}{r_2(x_k)}, \quad d_1 = q_1(\Delta x)^2, \quad d_2 = q_2(\Delta x)^2$$

where k is the number of the nodal points along the meridian, the stepsize being $\Delta x = l/N$, $l = L_s/R$.

The finite-difference equations (2.3) are written at all interior points, and two more equations are added for the boundary conditions at the boundary points with subscripts $k = 1$ and $k = N + 1$. The matrix coefficients of the internal equations are variables, occurring in which are the geometrical characteristics of the shell, which are computed for a given equation of the generator. The shape of the generator is usually given by an equation $r = f(z)$, where r is the radius and z is a coordinate, reckoned along the shell axis. Therefore, in order to compute the coefficients, one has to know the correspondence between the coordinates of a nodal point s_k along the meridian and z_k along the axis of revolution. In the general case this relation is established by a differential equation, the solution of which requires the solution of a Cauchy problem. In the simplest method for solving it, the coordinates z_k are computed by the formula

$$z_k = z_{k-1} + \frac{R\Delta x}{\sqrt{1 + (r'(z_{k-1}))^2}}, \quad k = 2, 3, \dots, N; \quad z_1 = 0 \tag{2.4}$$

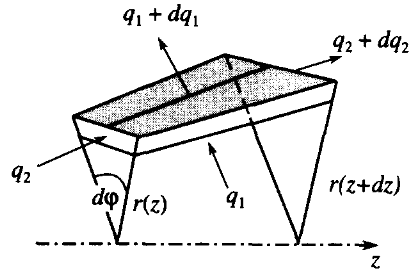

Fig. 3

The boundary conditions in the general case have the form

$$H_1\psi'(0) + G_1\psi(0) = e_1, \quad H_2\psi'(l) + G_2\psi(l) = e_2 \tag{2.5}$$

where H_i, G_i and e_i $(i = 1, 2)$ are the coefficient matrices and vectors of the right-hand sides of the given boundary conditions.

An approximation of the boundary conditions is introduced as follows:

$$H_1\left(\frac{\psi_2 - \psi_1}{\Delta x} + G_1\frac{\psi_2 + \psi_1}{2}\right) = e_1, \quad H_2\frac{\psi_{N+1} - \psi_N}{\Delta x} + G_2\frac{\psi_N + \psi_{N+1}}{2} = e_2 \tag{2.6}$$

The first boundary condition enables us to establish the relation between the vectors of the unknown functions at the first two nodal points

$$\begin{aligned} \psi_1 &= P_1\psi_2 + g_1, \quad P_1 = (2H_1 - G_1\Delta x)^{-1}(2H_1 + G_1\Delta x) \\ g_1 &= -2(2H_1 - G_1\Delta x)^{-1}e_1\Delta x \end{aligned} \tag{2.7}$$

Using the relation obtained, the first internal equation can be transformed to the same form, then the second and so on, up to and including the last equation

$$\begin{aligned} \psi_k &= P_k\psi_{k+1} + g_k, \quad P_k = (B - A^- P_{k-1})^{-1}A^+ \\ g_k &= (B - A^- P_{k-1})^{-1}(A^- g_{k-1} - d); \quad k = 2, 3, \ldots, N \end{aligned} \tag{2.8}$$

This procedure for computing the coefficients P_k of the transformed equations and vectors g_k by the recurrence formulae (2.8) is known as the forward sweep. After it has been implemented, the vector ψ_N can be eliminated from the second boundary condition and the vector of unknown functions computed at the boundary

$$\psi_{N+1} = (2H_2 + G_2\Delta x - (2H_2 - G_2\Delta x)P_N)^{-1}(2e_2\Delta x + (2H_2 - G_2\Delta x)g_N) \tag{2.9}$$

Now, in the cycle of reverse sweep, the transformed equations (2.8) are used to compute the vectors of the unknown functions at all nodal points. The stresses are then computed.

3. AN ALGORITHM FOR COMPUTING UNSTEADY TEMPERATURE FIELDS IN MERIDIONAL SECTIONS OF A SHELL

We will now consider the problem of computing axisymmetric temperature fields in a shell of revolution in the case of transient conditions of heat exchange at the outer and inner surfaces, varying in time and along the meridional coordinate s. For a coordinated computation of the temperature fields and thermal stresses, the heat-conduction equation should preferably be obtained in a curvilinear system of coordinates y, s, where y is the coordinate of the shell thickness directed along the outer normal, reckoned from the inner surface, and the coordinate s is directed along an arc of the meridian.

The heat-conduction equation can be obtained from the thermal balance condition of the small element shown in Fig. 3, which is cut out from the shell by two meridional sections, two circumferential surfaces and two equidistant surfaces. The equations of the equidistant surfaces may be represented as $y = c$ and $y = c + dy$, where $y = r - R_2(z)$, r is the polar coordinate in a circumferential section of the shell, R_2 is the radius of curvature of the interior surfaces, and dy is the distance between the surfaces and the thickness of the element.

In Fig. 3, q_1 and q_2 are the heat fluxes along the normal to the surfaces of the element and along an arc of the meridian s. The quantity of heat absorbed by the element in the time interval dt causes its temperature to change. Using Fourier's law of heat conduction, we can write this condition as an equation

$$\left\{-\lambda\frac{\partial T}{\partial y}r + \lambda\left(\frac{\partial T}{\partial y} + \frac{\partial^2 T}{\partial y^2}dy\right)(r + dy)\left(1 + \frac{dy}{R_1 + y}\right)\right\}d\varphi ds dt +$$

$$+\left\{-\lambda\frac{\partial T}{\partial s}r + \lambda\left(\frac{\partial T}{\partial s} + \frac{\partial^2 T}{\partial s^2}ds\right)\left(r + \frac{\partial r}{\partial s}ds\right)\right\}d\varphi dy dt = c\rho\frac{\partial T}{\partial t}r d\varphi ds dy dt$$

where

$$\frac{\partial r}{\partial s} = \frac{dR_2}{ds} = \operatorname{ctg}\theta\left(1 - \frac{R_2}{R_1}\right)$$

λ is the thermal conductivity, c and ρ are the specific heat and density of the shell material, and θ is the angle between the normal to the shell surface and the axis of revolution.

Collecting like terms and omitting terms of the fifth order of smallness, we obtain the equation

$$\frac{\partial^2 T}{\partial y^2} + \left(\frac{1}{R_1 + y} + \frac{1}{R_2 + y}\right)\frac{\partial T}{\partial y} + \frac{\partial^2 T}{\partial s^2} + \frac{\operatorname{ctg}\theta}{R_2 + y}\left(1 - \frac{R_2}{R_1}\right)\frac{\partial T}{\partial s} = \frac{c\rho}{\lambda}\frac{\partial T}{\partial t} \tag{3.1}$$

Attention must be drawn to the fact that the geometrical characteristics of the shell R_1, R_2 and θ, occurring in the coefficients of the equation just obtained, are determined, given the equation of the generator, as functions of the z coordinate, while the independent variable has been taken to be the s coordinate along an arc of the meridian. This does not introduce particular difficulties when one is solving the problem, and the z coordinate may be computed, for example, using formula (2.4).

For a numerical solution of the problem one can use the method of component-wise splitting of a differential operator with respect to the coordinates [7, p. 289], according to which, at each time step Δt, two boundary-value problems for one-dimensional differential equations are solved in succession:

$$\frac{c\rho}{\lambda}\frac{T^{n+i/2} - T^{n+(i-1)/2}}{\Delta t} = A_i\frac{T^{n+i/2} + T^{n+(i-1)/2}}{2}, \quad i = 1, 2; \; A_i = \frac{\partial^2}{\partial x_i^2} + \frac{\omega_i(z)}{r(z, y)}\frac{\partial}{\partial x_i}$$
$$\omega_1 = \left(1 + \frac{R_2 + y}{R_1 + y}\right), \; \omega_2 = \left(1 - \frac{R_2}{R_1}\right)\operatorname{ctg}\theta, \quad x_1 = y, \quad x_2 = s, \quad r(z, y) = R_2(z) + y \tag{3.2}$$

This method is quite natural in the context of heat-conduction problems, since it splits the heat flux process in a physical sense into two steps. The solution of the first differential equation corresponds to the propagation of heat in the direction of the x_1 coordinate only, and the solution of the second corresponds to propagation in the x_2 direction. Since these two processes are independent, it is very effective to separate the directions in this way. The method is still called the method of variable directions.

In order to eliminate possible skew in the solution of the problem, due to the fact that the same equation (the first) is solved at each time step, we can introduce a double-cycled splitting scheme, subsequently changing the order of solution of the one-dimensional boundary-value problems (3.2) ($n = 1, 2, \ldots, i = 1, 2$)

$$\frac{T^{n+i/2} - T^{n+(i-1)/2}}{\tau} = A_i(T^{n+i/2} + T^{n+(i-1)/2}), \quad \tau = \frac{\lambda\Delta t}{2c\rho} \tag{3.3}$$

$$\frac{T^{n+(i+2)/2} - T^{n+(i+1)/2}}{\tau} = A_{3-i}(T^{n+(i+2)/2} + T^{n+(i+1)/2}) \tag{3.4}$$

Note that when the double-cycle scheme is used, there are two time steps for each index n. At the first step one solves Eqs (3.3) first with the operator A_1 and then with the operator A_2. At the second step, Eqs (3.4) are solved, first with the operator A_2 for $i = 1$, and then, for $i = 2$, with the operator A_1. This alternation of two schemes excludes the possibility that either of the two equations will predominate.

The core of the algorithm. The solution of the problem using the double-cycle splitting scheme involves, at each step of the computations, the solution of a one-dimensional boundary-value problem in four cycles; this may be represented in a generalized form as one standard finite-difference solution procedure using scalar double-sweep for a fairly simple differential equation

$$A(u+u^0) = \frac{u-u^0}{\tau} \tag{3.5}$$

where u^0 is the solution computed at the previous step, $u(x)$ is the solution sought at the next step, and A is a differential operator in one of two directions.

Let us proceed to the finite-difference analogue of Eq. (3.5)

$$\begin{gathered} \frac{u_{k-1}-2u_k+u_{k+1}}{(\Delta x)^2}+\frac{\beta_k}{r_k}\frac{u_{k+1}-u_{k-1}}{2\Delta x}-\frac{1}{\tau}u_k = q_k, \quad k = 2, 3, \dots, N \\ q_k = -\left(\frac{1}{\tau}u_k^0+\frac{u_{k-1}^0-2u_k^0+u_{k+1}^0}{\Delta x^2}+\frac{\beta_k}{r_k}\frac{u_{k+1}^0-u_{k-1}^0}{2\Delta x}\right) \end{gathered} \tag{3.6}$$

where k is the number of the nodal points with coordinates $x_k = (k-1)\Delta x$ $(k = 1, 2, \dots, N+1)$, $\Delta x = l/N$ and l is the dimension of the given domain in one of the directions under consideration. The number of nodal points $N+1$ in each direction may be specified in different ways. In the differential operators A_1 and A_2 the coefficients β_k are equal to $\omega_1(z_k)$ and $\omega_2(z_k)$, respectively.

The system of equations (3.6) results when the heat-conduction equation is written at all interior points. Each equation corresponds to the thermal balance condition for a layer of thickness Δx. Allowance has yet to be made for the specific heat of the two layers adjoining the boundaries, the thickness of each of which is $\Delta x/2$. Hence, the boundary conditions must be formulated allowing for the specific heat of these layers adjoining the boundaries. Let us consider, for example, the convective heat exchange condition on the outer surface of the shell, which, when the problem is solved analytically, at $x_1 = h$ has the form

$$-\lambda\frac{\partial u}{\partial x}+h_w(x_2,t)(T_w(x_2,t)-u(h)) = 0 \tag{3.7}$$

where $h_w(x_2, t)$ is the heat transfer coefficient and $T_w(x_2, t)$ is the temperature of the boundary layer of the environment; in the general case, both of these are functions of the meridional coordinate x_2 and the time t.

This condition means that the external and internal heat fluxes passing through the surface $x = h$ are equal. When this condition is expressed in terms of finite differences, it must be taken into consideration that the external flux passes through the surface $x = h$, but the interior flux goes through the second surface of the adjoining layer $x = h - \Delta x/2$, and part of the heat is absorbed by the layer adjoining the boundary. Taking the specific heat of the boundary layer into account, Eq. (3.7), expressed in terms of finite differences, takes the form

$$-\lambda\frac{u_{N+1}-u_N}{\Delta x}+h_w(T_w-u_{N+1}) = c\rho\frac{\Delta x}{2}\frac{u_{N+3/4}-u_{N+3/4}^0}{\Delta t}, \quad u_{N+3/4} = \frac{3u_{N+1}+u_N}{4} \tag{3.8}$$

As a second example, let us write the condition of thermal insulation of the inner surface

$$\lambda\frac{u_2-u_1}{\Delta x} = c\rho\frac{\Delta x}{2}\frac{u_{1+1/4}-u_{1+1/4}^0}{\Delta t}, \quad u_{1+1/4} = \frac{3u_1+u_2}{4} \tag{3.9}$$

Thus, the system of finite-difference equations (3.6), together with the boundary conditions, may be represented in the general case in the form

$$
\begin{aligned}
&-B_1u_1+A_1^+u_2 = d_1 \quad A_k^-u_{k-1}-B_ku_k+A_k^+u_{k+1} = d_k, \quad k = 2,3,\dots,N \\
&A_{N+1}^-u_N-B_{N+1}u_{N+1} = d_{N+1}
\end{aligned}
\tag{3.10}
$$

where the coefficients of the inner equations are

$$
A_k^{\pm} = 1\pm\frac{\beta_k\Delta x}{2r_k}, \quad B_k = 2+\frac{2c\rho(\Delta x)^2}{\lambda\Delta t}, \quad d_k = (\Delta x)^2q_k \tag{3.11}
$$

The coefficients of the boundary conditions depend on the parameters of the given heat exchange conditions. In examples (3.8) and (3.9) shown above

$$
\begin{aligned}
&B_1 = 1+\frac{3(\Delta x)^2}{16\tau}, \quad A_1^+ = A_{N+1}^- = 1-\frac{(\Delta x)^2}{16\tau}, \quad d_1 = -\frac{(\Delta x)^2}{16\tau}(3u_1^0+u_2^0) \\
&B_{N+1} = B_1+\frac{h_w\Delta x}{\lambda}, \quad d_{N+1} = -\frac{(\Delta x)^2}{16\tau}(3u_{N+1}^0+u_N^0)-\frac{h_w\Delta x}{\lambda}T_w
\end{aligned}
\tag{3.12}
$$

The coefficient matrix of the system of algebraic equations (3.10) is tridiagonal, and the solution may be obtained by double-sweep. In the forward sweep cycle, all the inner equations are transformed to a two-term form

$$
u_k = P_ku_{k+1}+g_k, \quad k = 2,3,\dots,N \tag{3.13}
$$

The first boundary condition yields $P_1 = A_1^-/B_1, g_1 = -d_1/B_1$, while the coefficients of the inner equations are evaluated by recurrence formulae

$$
P_k = (B_k-A_k^-P_{k-1})^{-1}A_k^+, \quad g_k = (B_k-A_k^-P_{k-1})^{-1}(A_k^-g_{k-1}-d), \quad k = 2,3,\dots,N \tag{3.14}
$$

After carrying out these transformations in the second boundary layer, we can eliminate $u_N = P_Nu_{N+1} + g_N$ and evaluate the temperature at the boundary

$$
u_{N+1} = \frac{d_{N+1}-A_{N+1}^-g_N}{A_{N+1}^-P_N-B_{N+1}} \tag{3.15}
$$

Then, using Eqs (3.13) in the reverse order, we can evaluate the temperature at all nodal points.

In order to solve the problem as a whole, one has to represent the temperatures at the nodal points of the mesh as a two-dimensional array. At the beginning of the computation of this array $T(I, J)$, the values of the initial temperatures T_{ij}^0 ($i = 1, 2, \dots, N_1 + 1; j = 1, 2, \dots, N_2 + 1$) are filled in.

Next, in order to proceed to the next instant of time, the double-sweep procedure (3.13)–(3.15) is implemented with respect to the coordinate $x = x_1$ for successive values of $j = 2, 3, \dots, N_2$, and then with respect to the coordinate x_2 for all successive values $i = 2, 3, \dots, N_1$.

To clarify this account, we present the beginning of these computation in greater detail.

We fix $j = 2$ and carry out the following operations:

(1) transport one vector of the array $T(I, J)$ to a one-dimensional Fourier array and write this condition in the form of an equation

$$
U(I) = T(I,J), \qquad J = 2, \qquad I = 1,2,\dots,N_1+1
$$

(2) compute the coefficients A_i, B_i, C_i, d_i by formulae (3.11) and (3.12) for $i = 1, 2, \dots, N_1 + 1$; $\Delta x = \Delta x_1$.

(3) using the standard procedure (3.13)–(3.15) to solve the one dimensional problem, compute the new temperature values u_i of the vector $U(I)$. Place the vector thus obtained in the two-dimensional array T at the position used in this procedure; in the present case it is the vector with index $j = 2$ (the second column of the array).

(4) change the index j by one and repeat the computations described above, and so on up to $j = N_2$.

Now, continuing in analogous fashion, carry out sweeps with respect to the x_2 coordinate. After sweeping along and across the array T, the temperature values will have been filled in at all the nodal points, and thus one further time step will have been completed. Proceeding to the next instant of time, sweeps are carried out in reverse sequence – first with respect to x_2 and then with respect to x_1. As a result the whole time sequence consists of pairs of steps.

It should be observed that sweeps according to the standard procedure (3.13)–(3.15) are only carried out for the interior coordinates of the curves, and therefore the temperatures at the nodal points are evaluated by extrapolating the temperature at adjacent points.

We have thus presented algorithms for solving one class of transient boundary-value problems in thermoelasticity, which may find wide application in engineering practice for computing temperature fields and thermal stresses and for parametric optimization of composite shell structures.

The possibility of extending scalar double-sweep to the method of matrix double-sweep along the lines of the scheme presented here was first published by the present author, as applied to the solution of stability problems for shells [2]. It should be observed that the method of matrix double-sweep is not always trivial. An investigation of the special features of the double-sweep method in solving problems of the stability of shells may be found, e.g. in [3]. Non-trivial situations for the application of matrix double-sweep also arise in the finite-difference approximation of boundary conditions when all the conditions at the boundary are given for one part of the vector of unknown functions, and at the other boundary for the other part of the vector, such as, e.g. in solving boundary-value problems for the equations of heat transfer by radiation, which must be taken into consideration when approximating differential equations by a difference scheme [4].

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